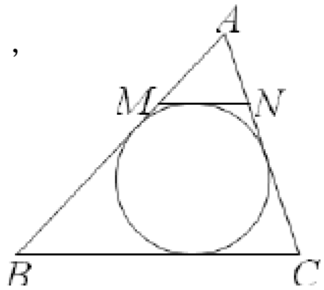
12.如右圖，已知ΔABC的周長為 s，在$\overline{AB}$，$\overline{AC}$上分別取點 M，N，

使得$\overline{MN} // \overline{BC}$，且$\overline{MN}$與$\Delta ABC$的內切圓相切，則$\overline{MN}$的最大值

為______。【100.中正高中二招 ★☆☆三角函數正餘弦定理】



【解】:

設ΔABC的內切圓半徑為 r、其面積為Δ，且$\overline{BC}=a$，$\overline{BC}$上的高為 h

則$\Delta=\frac{1}{2}rs \Rightarrow r=\frac{2\Delta}{s}=\frac{ah}{s}\Rightarrow 2r=\frac{2ah}{s}$

$\because \Delta AMN \sim \Delta ABC$，$\therefore \frac{\overline{MN}}{\overline{BC}}=t \Rightarrow \overline{MN}=ta$，且$\Delta AMN$的高為$th$，則

$2r=h-th=(1-t)h \Rightarrow \frac{2ah}{s}=(1-t)h \Rightarrow a=\frac{1}{2}s(1-t)\Rightarrow at=-\frac{1}{2}s(t-\frac{1}{2})+\frac{s}{8}$

故當$t=\frac{1}{2}$時，則$\overline{MN}$的最大值為$\frac{s}{8}$

16.直角ΔABC中，$\angle C$是直角，設ΔABC的內切圓 O 交$\overline{AB}$於 D，r 為內切圓 O 的半徑，$\overline{AB}=c$，求證：【100.中正高中二招 ★★☆和角公式及倍半角公式】

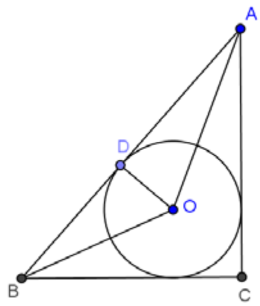
(1) $r=\dfrac{c}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}$

(2) $r\le\frac{c}{2}(\sqrt{2}-1)$

【解】:

(1)如右圖所示，則$\overline{OA}$、$\overline{OB}$分別為為$\angle A$、$\angle B$的分角線，

圓 O 與$\overline{AB}$切於 D 點，則$\overline{OD}\perp\overline{AB}$，且$\overline{OD}=r$



由$\cot\frac{A}{2}=\frac{\overline{AD}}{r}\Rightarrow \overline{AD}=r\cot\frac{A}{2}$

由$\cot\frac{B}{2}=\frac{\overline{BD}}{r}\Rightarrow \overline{BD}=r\cot\frac{B}{2}=r\cot(45^\circ-\frac{A}{2})$

$\therefore c=\overline{AD}+\overline{BD}=r(\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2}))\Rightarrow r=\dfrac{c}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}$

(2) $\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\tan(45^\circ+\frac{A}{2})=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\dfrac{\tan 45^\circ+\tan\frac{A}{2}}{1-\tan 45^\circ\tan\frac{A}{2}}$

$$=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}=\frac{\cos\frac{A}{2}(\cos\frac{A}{2}-\sin\frac{A}{2})+\sin\frac{A}{2}(\cos\frac{A}{2}+\sin\frac{A}{2})}{\sin\frac{A}{2}(\cos\frac{A}{2}-\sin\frac{A}{2})}$$

$$=\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}-\sin^2\frac{A}{2}}=\frac{1}{\frac{1}{2}\sin A-\frac{1-\cos A}{2}}=\frac{2}{\sin A+\cos A-1}=\frac{2}{\sqrt{2}\sin(A+45^\circ)-1}$$

$\because 0<A<\frac{\pi}{2}$，$\therefore \frac{\pi}{4}<A+45^\circ<\frac{3\pi}{4}$，故 $\frac{\sqrt{2}}{2}<\sin(A+45^\circ)\le 1$

$\because \frac{1}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}=\frac{\sqrt{2}\sin(A+45^\circ)-1}{2}$，$\therefore \frac{1}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}\le\frac{\sqrt{2}-1}{2}$

$\therefore \frac{c}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}\le\frac{c(\sqrt{2}-1)}{2}\Rightarrow r\le\frac{c(\sqrt{2}-1)}{2}$